# Comparison of aperiodic stochastic resonance in a bistable system realized by adding noise and by tuning system parameters

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Two methods of realizing aperiodic stochastic resonance (ASR) by adding noise and tuning system parameters in a bistable system, after a scale transformation, can be compared in a real parameter space. In this space, the resonance point of ASR via adding noise denotes the extremum of a line segment, whereas the method of tuning system parameters presents the extrema of a parameter plane. We demonstrate that, in terms of the system performance, the method of tuning system parameters takes the precedence of the approach of adding noise for an adjustable bistable system. Besides, adding noise can be viewed as a specific case of tuning system parameters. Further research shows that the optimal system found by tuning system parameters may be subthreshold or suprathreshold, and the conventional ASR effects might not occur in some suprathreshold optimal systems.

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## I. INTRODUCTION

The concept of stochastic resonance (SR) was originally put forward in two seminal papers by Benzi and collaborators [1,2], in which the response of a nonlinear system to a weak periodic input signal is amplified by adding an optimal level of additive noise [3,4]. Such a nonlinear system has a continuum power spectrum owing to the input stochastic force, i.e., noise [1]. This counterintuitive phenomenon has then attracted much attention in the past two decades, and recently on the transmission of actual information-carrying signals via aperiodic stochastic resonance (ASR) [4–14]. Since the introduction of SR, the main body of literature devoted to the investigation of SR-type phenomenon by the method of adding noise [1–31], what we call the conventional SR in this paper. However, recent studies suggested that the strategy of conventional SR seems to be sub-optimal in terms of information flow, and only provides a positive function role of noise for subthreshold signals [13–15,26–28]. This leads to a series of discussions, especially on the incorporation of conventional SR into the neuronal information processing [15,26–31].

Benzi [1] has suggested that the above SR phenomenon was the cooperation of the stochastic system and the input signal. According to this view of Ref. [1], adding noise is tuning the stochastic system parameter in essence. There is another method to tune the stochastic system, this is, directly adjusting system parameters. It seems the usual dynamical resonance by adjusting the inherent frequency of the system to the frequency of an external periodic force. Thus, one natural generalization is to research the cooperative effect in an adjustable system subject to a given mixture of the external time-dependent force and the noise [25]. The system parameter is then considered as an important tunable element for the study of SR and ASR [32–42]. It was demonstrated that there are optimal threshold values in the detection of noisy signals with neuronlike threshold crossing detectors in the context of SR [32]. In a review paper [33], the fact that SR can be fulfilled by adjusting system parameters was emphasized in the signal processing field. Adaptively selecting the parameters of SR devices was investigated for the short record detection [34] and as the low power detection algorithm [35]. This approach was also introduced as the natural background for the design stage of nonlinear measuring devices [36]. Recently, schemes of how to choose an optimal stochastic resonator were developed in the presence of some non-Gaussian noises [37,38]. Moreover, SR can be controlled so as to either suppress or enhance the output power at the signal frequency by sinusoidally modulating the barrier height between the two wells of bistable systems [39,40]. Additionally, the SR-type phenomena via tuning system parameters were researched in the multi-frequency signal processing [41] and the binary signal transmission [42].

Thus, two problems arise: this is, what is the relationship between two approaches of adding noise and tuning the system parameters, and can "ASR" realized by tuning system parameters refer to the concept of conventional ASR via adding noise?

In this paper, we compare two approaches of realizing ASR effects via adding noise and tuning system parameters

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in a bistable system, theoretically and numerically. Since the input is assumed as random binary signals, the system performance is quantified by an information measure of the bit error rate (BER). Thus, the conventional ASR phenomenon studied in this Letter is such an effect that, for an initially adopted system, the BER presents a minimal value at an optimal nonzero noise intensity. Given a noisy input, as will be observed, the BER is a nonmonotone function of system parameters, which is the resonance effect realized by tuning system parameters. In Sec. II, the approach of adding noise, after a scale transformation, can be viewed as a specific case of the method of tuning system parameters. The conventional ASR can be mapped into an optimal problem of a line segment, whereas the method of tuning system parameters provides the optimal solutions of a real parameter plane. In terms of the system performance, the method of tuning system parameters is more applicable than the adding noise approach. In Sec. III, the optimal systems found by the approach of tuning system parameters are classified as subthreshold and suprathreshold. We suggest that the subthreshold optimal systems can be viewed as the conventional aperiodic stochastic resonators. There is a new form of ASR, i.e., the residual ASR phenomenon, surviving in the slightly suprathreshold optimal systems [45]. In other suprathreshold optimal systems, no conventional ASR effects occur. Then, these suprathreshold optimal systems can be considered as the results of the nonlinear system optimization. Finally, we argue that our conclusions of the comparison between the two methods can be extended to other nonlinear systems studied in the context of SR or ASR phenomenon.

## II. THE RELATIONSHIP BETWEEN THE ASR BY ADDING NOISE AND THAT VIA TUNING SYSTEM PARAMETERS

### A. Fundamental model and theory

Consider a nonlinear dynamic system whose internal state $x(t)$ evolves according to [6],

$$\tau_a \frac{dx(t)}{dt} = x(t) - \frac{x^3(t)}{X_b^2} + s(t) + \eta(t), \quad (1)$$

where system parameters $\tau_a > 0$ and $X_b > 0$. The input information-bearing signal is $s(t)$, and $\eta(t)$ is an additive Gaussian white noise with autocorrelation $\langle \eta(t)\eta(0)\rangle = 2D\delta(t)$ and zero-mean. Here, $D$ denotes the noise intensity. In the absence of input signal, the symmetrical bistable potential is $V(x) = -x^2/2 + x^4/(4X_b^2)$ with potential barrier of $\Delta V = X_b^2/4$ and the potential minima are just located at $\pm X_b$. In the presence of an input signal, the potential becomes $V(x) = -x^2/2 + x^4/(4X_b^2) - s(t)x$, and each potential minimum is raised or lowered relative to the barrier height [6]. In this paper, the input signal is assumed as the baseband binary pulse amplitude modulated signals, which transmit digit 0 by a waveform $s_0(t) = -A$ or digit 1 by a waveform $s_1(t) = +A$ in a time interval $[0\ T_B]$. Here, $A$ is the signal level and $T_b$ is called bit duration. Since the source digits are encoded according to the messages, digits 0 and 1 are often random in a sequence and the input binary signal $s(t)$ consists of the corresponding randomly arranged waveforms $s_0(t)$ and $s_1(t)$. $s(t)$ is then an aperiodic information-bearing binary signal. The input signal-to-noise ration (SNR) per bit defined as $\varepsilon_b/(4D)$ is appropriate for measuring this input noisy binary signal $s(t)$, where $\varepsilon_b = \int_0^{T_b} s_i^2(t)dt$ for $i = 0, 1$ [46].

In each bit duration of $T_b$, the system is driven by a constant signal, i.e., $s_0(t) = -A$ or $s_1(t) = +A$, with the additive noise $\eta(t)$. On this condition, the corresponding Fokker–Planck equation to Eq. (1) is given by

$$\tau_a \frac{\partial \rho(x,t)}{\partial t} = \left[ \frac{\partial}{\partial x} V'(x,A) + \frac{D}{\tau_a} \frac{\partial^2}{\partial x^2} \right] \rho(x,t), \quad (2)$$

where $V'(x) = -x + x^3/(X_b^2) \pm A$ and $\rho(x,t)$ obeys the natural boundary conditions that it vanishes at large $x$ for any $t$ [47]. The steady state solution of Eq. (2) reads [47] as

$$\rho(x|s_i(t)) = \lim_{t\to\infty} \rho(x,t) = C \exp\left[ -\frac{\tau_a V(x)}{D} \right], \quad (3)$$

where $C$ is the normalization constant and $V(x) = -x^2/2 + x^4/(4X_b^2) \pm Ax$.

This steady state solution, i.e., Eq. (3), is suitable for describing the probability density of the system output if the input signal $s(t)$ keeps a constant amplitude as $t \to \infty$. However, the information-bearing input $s(t)$ might take different amplitudes in successive bit durations, otherwise $s(t)$ has few information contents. Hence, the probability density of the system output is explicitly nonstationary. In the presence of noise, if the system is modulated by a current input waveform $s_i(t)$ for $i = 0, 1$, a nonstationary probability density model of the system output is established as [42]

$$\rho(x,t|s_i(t)) = N \exp\left[ -\frac{\tau_a G(x,\alpha)}{D} \right], \quad (4)$$

where $N$ is the normalization constant, $G(x,\alpha) = -x^2/2 + x^4/(4X_b^2) \pm \alpha A x$, $\alpha = 1 - \exp[-\lambda_1 t]$ and $\lambda_1$ is the system response speed introduced in Appendix A. As $\lambda_1 t \to +\infty$, $\rho(x,t|s_i(t))$ degenerates into the stationary densities $\rho(x|s_i(t))$ of Eq. (3).

In this nonstationary process, erroneous bit symbols are measured by the bit error rate (BER), and its minimal value corresponds to the maximum information transfer between the system input–output. First, we introduce a decode scheme from the observation of the system output $x(t)$: Sample $x(t)$ at the end time of each bit duration $T_b$, a successive sampled values, $x(jT_b)$ for $j = 1, 2, \ldots$, are then obtained. By comparing $x(jT_b)$ with the decision threshold $l$, the recovered binary digit reads 1 if $x(jT_b) > l$, otherwise it is 0. Then, the probability of error $P(0|1)$ denotes that the recovered digit is decoded as 0 while the input source digit is 1. The other converse case is represented as the probability of error $P(1|0)$. Thus, the total probabilities of error, i.e., the BER, is

$$P_e = P(1)P(0|1) + P(0)P(1|0), \quad (5)$$

where $P(1)$ and $P(0)$ represent the probabilities of digits 1 and 0 in a sequence, respectively. We further assume that the source input digits occur with equiprobabilities, i.e., $P(1) = P(0) = 0.5$, and are statistically independent. Thus, the system of Eq. (1) with input binary digits and output binary readings, can be viewed as an information channel transmitting binary data. It has been analyzed as a memoryless symmetric binary channel in Refs. [6,42], with the decision threshold $\ell = 0$. A theoretical expression of the BER then takes the form as

$$P_e = \frac{1}{2}[P(1|0) + P(0|1)]$$
$$= \frac{1}{2}\left[\int_{-\infty}^{0} \rho(x, T_b|s_1(t))dx + \int_{0}^{+\infty} \rho(x, T_b|s_0(t))dx\right]. \quad (6)$$

Moreover, a block scheme was designed for transmitting binary signals via the bistable system of Eq. (1) in Ref. [42], wherein the input signal is generated by a pseudorandom binary signal generator. Next, we will compare the two methods of adding noise and tuning system parameters with the above theory and the simulation scheme at hand.

### B. Theoretical comparison of ASR effects realized by two approaches of adding noise and tuning system parameters

In this subsection, we interpret two approaches of adding noise and tuning system parameters geometrically. Rescale the variables as

$$t = T_b\tau, \quad x = \sqrt{D/T_b}y, \quad X_b = \sqrt{D/T_b}\overline{X_b},$$
$$A = \sqrt{D/T_b}\overline{A}, \quad \tau_a = T_b\overline{\tau_a}, \quad (7)$$

Eq. (1) becomes

$$\overline{\tau_a}\frac{dy(\tau)}{d\tau} = y(\tau) - \frac{y^3(\tau)}{\overline{X_b^2}} \pm \overline{A} + \xi(\tau), \quad (8)$$

where $\langle\xi(\tau)\xi(0)\rangle = 2\delta(\tau)$ and $\overline{A}^2/4$ is just the input SNR per bit $\varepsilon_b/(4D)$. Usually, in order to investigate the conventional ASR via adding noise, the system parameters $\tau_a$, $X_b$, the signal amplitude $A$ and the bit duration $T_b$ are fixed in an initially adopted system. Thus, increasing the noise intensity $D$ represents decreasing the transformed variables $\overline{X_b}$ and $\overline{A}$. This indicates that, after this transformation of Eq. (7), the conventional ASR effects realized by adding noise can be explained as optimizing the system parameter $\overline{X_b}$ with the degraded input $\overline{A}$.

Figure 1 shows the real parameter space $S:\{\overline{\tau_a}, \overline{X_b}, \overline{A}\}$, wherein the point **a** denotes the initial adopted system parameters $\overline{\tau_a}$ and $\overline{X_b}$ and the corresponding input $\overline{A_0}$. Here, $\overline{A_0^2}/4 = \varepsilon_b/(4D_0)$ and $D_0$ is the initial input noise intensity. As $D$ increases, $\overline{X_b}$ and $\overline{A}$ will decrease, but $\overline{\tau_a}$ keeps invariable. When the amount of noise is added appropriately, a minimal value of the BER is obtained and the conventional ASR phenomenon occurs. This optimal noise intensity corresponds to the resonance point ***b*** in Fig. 1. A large amount of input noise denotes the points at the extension line of the segment ***ab***, such as the point ***c***. Thus, adding noise forms a line segment in this parameter space $S$ and the resonance point of the conventional ASR can be mapped into an extremal solution along this line segment.

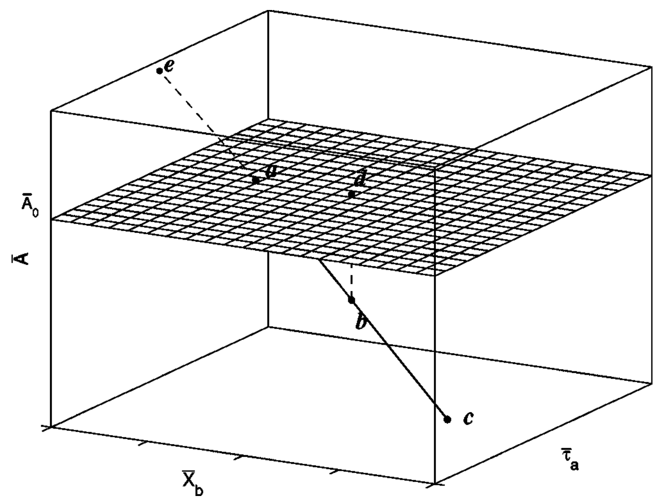


FIG. 1. The real parameter space $S:\{\overline{\tau_a}, \overline{X_b}, \overline{A}\}$. The conventional ASR via adding noise is mapped onto an optimal problem of the line segment ***ac***. The method of tuning parameters searches the optimal pair(s) of system parameters $\overline{\tau_a}$ and $\overline{X_b}$ in the plane of $\overline{A_0}$.

In contrast to the conventional ASR, the other method is tuning system parameters $\tau_a$ and $X_b$ for a given noisy input ($A$, $T_b$ and $D = D_0$ are fixed). This method is an optimal problem in the real parameter plane of $\overline{A_0}$, as illustrated in Fig. 1. It is demonstrated in Appendix B that, for the selected system with the fixed system parameters, the BER will increase as the input $\overline{A}$ degrades. Thus, we can deduce that the point ***d*** in the parameter plane of $\overline{A_0}$ will provide a lower value of the BER than the conventional resonance point ***b***. Here, ***d*** is the projective point of the point ***b*** in the plane of $\overline{A_0}$ with the same system parameters $\overline{\tau_a}$ and $\overline{X_b}$. Moreover, the method of tuning system parameters finds the optimal pair(s) of parameters $\overline{\tau_a}$ and $\overline{X_b}$ in the whole parameter plane of $\overline{A_0}$, which must be better than the point ***d***.

Furthermore, there is another difficult case confronted by the approach of adding noise, this is the resonance point ***e*** is with an input $\overline{A}$ higher than the initially given $\overline{A_0}$ (see Fig. 1). In this case, the initial input noise intensity $D_0$ is beyond the optimal one corresponding to the conventional resonance point ***e***. Therefore, the method of adding noise cannot be utilized to realize any conventional ASR effects. It would be practical to adjust system parameters to obtain the corresponding minimal value of the BER.

Consequently, in view of the system performance BER, we conclude that the method of tuning system parameters always outperforms the adding noise approach. This theoretical analysis will be manifestly confirmed by the following numerical results.

### C. Numerical comparison results of ASR effects realized by two approaches

In this subsection, we will numerically demonstrate the conclusion of the comparison between two approaches of adding noise and tuning system parameters. We have numerically integrated the stochastic differential equation of Eq. (1) using an Euler-Maruyama discretization method with a small sampling time step $\Delta t \ll \tau_a$ [43]. The block scheme for transmitting binary data by this nonlinear system of Eq. (1) has been designed in Ref. [42]. With this designed block scheme, the BER can be automatically recorded in numerical simulations.

A simulation example of the conventional ASR phenomenon is shown in Fig. 2(A). The initially adopted system is with parameters $\tau_a = 10^{-0.9}$ and $X_b = 10^{0.9}$. Here we take $D_0 = 0.285$, $A=3$, and $T_b = 4$ s. Then the input SNR per bit is 15 dB, i.e., the point $\boldsymbol{a}$ in Fig. 2(A). The conventional ASR effect can be realized by adding an appropriate amount of noise, resulting in the resonance point $\boldsymbol{b}$ ($D_1 = 0.84$). The corresponding BER is then given as $3.1 \times 10^{-3}$ and the input SNR per bit is degraded to 10.3 dB. However, if the initially given input SNR per bit is 5 dB at the point $\boldsymbol{c}$ ($D_2 = 2.846$) and beyond the conventional ASR resonance point $\boldsymbol{b}$ ($D_1 = 0.84$), it will be of no use to improve the system performance by adding noise. This case may be often met in a nonstationary noisy environment. Moreover, another difficulty of controlling noise is that the optimal amount of noise to be added is highly dependent on the *a priori* statistical characteristics of noise.

Figures 2(B) and 2(C) present the simulation results of the method of tuning system parameters at the given input SNRs per bit of 10.3 dB and 5 dB, respectively. For comparison, the points $\boldsymbol{b}$ and $\boldsymbol{c}$ of the conventional ASR are also plotted. For the given SNR per bit 10.3 dB, the numerical results of tuning system parameters, shown in Fig. 2(B), can give a much-improved BER of $1.02 \times 10^{-4}$ (i.e., the point $b_1$). It is shown in Fig. 2(C) that, at the input SNR per bit of 5 dB, the minimal value of the BER is obtained as $1.603 \times 10^{-2}$ by optimally adjusting the system parameters (see the point $c_1$), whereas the method of adding noise only provides the BER of $1.1 \times 10^{-1}$. Thus, for a given noisy signal, we can tune the system parameters, rather than adding noise to the initially selected system, to obtain a corresponding minimal value of the BER. These numerical illustrative comparisons of Fig. 2 demonstrate that the method of tuning parameters is more applicable for the adjustable bistable system, confirming the theoretical analysis in this subsection.

Many researchers have observed SR or ASR effects in biology systems and neuroscience [3]. But no living organism can control the noise structure of the environment [22]. A rising problem is whether the biology system tunes itself to adapt to the noisy environment or uses the internal noise via SR, or both. Some pioneering studies suggested that SR or ASR may be useful if there was insufficient adaptability in sensory systems [15,26], and in terms of recent results of the suprathreshold SR [13–15,26], the neuronal noise can have a positive beneficial role regardless of stimulus intensity or the adaptive capability of neurons. Further concrete investigations on this question are meaningful.

## III. THE PARAMETER TUNING RESONANCE PHENOMENA IN SUBTHRESHOLD AND SUPRATHRESHOLD REGIONS

In the context of the conventional ASR, the input signal is subthreshold that no deterministic switching can occur in the presence of the signal alone [19]. When $A \geq 2X_b/\sqrt{27}$ [6,19], the system bistability is destroyed and the input signal becomes suprathreshold [14,19]. In this Letter, the parameter $X_b$ is adjustable and the input signal level $A$ is fixed. This yields

$$X_b^c = \sqrt{27}A/2, \tag{9}$$

by which the system under study is classified as subthreshold ($X_b > X_b^c$) and suprathreshold ($X_b \leq X_b^c$).

In practice, the adjustable parameters $\tau_a$ and $X_b$ are often restricted in some regions, and $X_b$ may take the value in the region of $X_b > X_b^c$ or $X_b \leq X_b^c$. For a given noisy input, the methodology of tuning system parameters is as follows: Choose the parameter $X_b$, and then deduce the corresponding optimal parameter $\tau_a$ such that the BER is minimal. The optimal system, with this optimal pair of parameters $X_b$ and $\tau_a$, may be subthreshold or suprathreshold in respect of the parameter $X_b$. It is worthy of note that the optimal systems found by tuning system parameters, shown in Fig. 3, are just corresponding to the valley bottom of the BER throughout the parameter plane of $X_b$ and $\tau_a$ in Fig. 2(C).

There is then an interesting question we can ask; this is, what kind of role does the noise play in these optimal systems? Furthermore, according to the role of noise, we will attempt to clarify these optimal systems searched by the method of tuning system parameters.

Figure 4 shows the performances of these optimal systems of Fig. 3 as the function of the input noise intensity $D$, wherein the role of the noise is revealed. We give the main conclusions of our study as follows.

(i) For the subthreshold optimal systems with $X_b > X_b^c \approx 10^{0.892}$, Fig. 4 clearly displays their conventional ASR type behaviors (i.e., the BER presents a minimal value at a nonzero level of noise). It indicates that the noise plays the constructive role in these subthreshold systems. In other words, they can be looked as the conventional ASR systems. In terms of this explanation, we can also refer the parameter-tuning resonance phenomena to the concept of conventional ASR for the subthreshold optimal systems.

(ii) When $X_b \leq X_b^c$, the optimal systems become suprathreshold. Figure 4 shows that the conventional ASR phenomenon survives until $X_b$ is smaller than but close to the dynamical threshold $X_b^d$. This result is in keeping with the analyses of Refs. [44,48] and $X_b^d$ can be evaluated as $X_b^c/1.3$ for this kind of waveform of $s(t)$ [44]. This new form of ASR effects, termed the residual ASR, was studied in our recent work [45] in detail. We see that the positive role of the noise extends to this slightly supratheshold region as $X_b^d - \epsilon < X_b < X_b^c$, where $\epsilon$ is a quite small value.

(iii) As $X_b < X_b^d - \epsilon$, the BER, illustrated in Fig. 4, becomes a monotonic function of the noise intensity in these suprathreshold optimal systems, and the positive role of noise disappears in this dynamical process. Thus, we argue that, in this "very" suprathreshold range of $X_b < X_b^d - \epsilon$, these suprathreshold optimal systems should be considered as the results of the *nonlinear system optimization*.

Consequently, the optimal systems found by tuning parameters can be classified as subthreshold as $X_b > X_b^c$ and

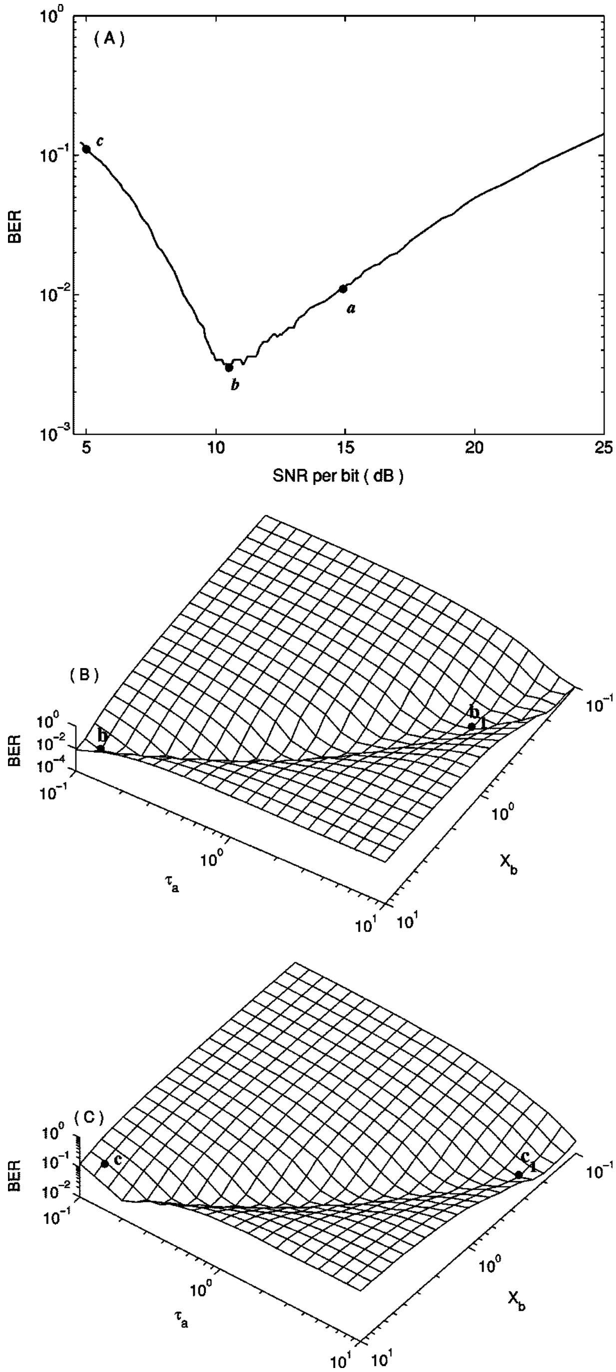


FIG. 2. Numerical results of the BER obtained by two methods of adding noise and tuning system parameters. Here, $A=3$, $T_b=4$ and $\Delta t=0.01\tau_a$. (A) A simulation example of the conventional ASR effect via adding noise. The system parameters $\tau_a=10^{-0.9}$ and $X_b=10^{0.9}$. The initial point ***a***, the conventional resonance point ***b*** and the point ***c*** are corresponding to the input SNR per bit of 15 dB ($D_0=0.285$), 10.3 dB ($D_1=0.84$), and 5.0 dB ($D_2=2.846$), respectively; (B) a plot of the BER versus parameters $\tau_a$ and $X_b$ at the given input SNR per bit of 10.3 dB. The conventional resonance point ***b*** is also reflected in this surface, and the method of tuning system parameters can provide a much lower BER at the the point $\boldsymbol{b}_1$; (C) a plot of the BER versus parameters $\tau_a$ and $X_b$ at the given input SNR per bit of 5.0 dB. Clearly, the point $\boldsymbol{c}_1$ is better than the point ***c*** in terms of the system performance.

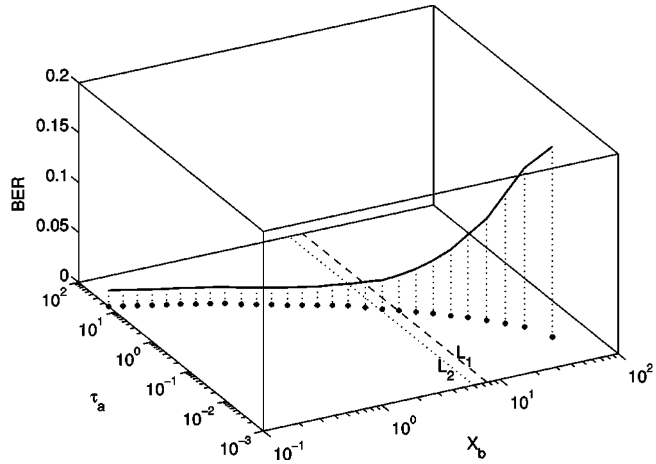


FIG. 3. As the system parameter $X_b$ takes different values, the corresponding optimal system parameter $\tau_a$ with the minimal BER. Note that these optimal pairs of system parameters are just corresponding to the valley bottom of the BER of Fig. 2(C). Here, $A=3$, $T_b=4$s, $\Delta t=0.01\tau_a$, and $D=2.846$. The optimal systems are divided into subthreshold and suprathreshold regions by the line of $L_1$ ($X_b^c=\sqrt{27}A/2\approx 10^{0.892}$). The dynamical threshold is represented by the line $L_2$ ($X_b^d=X_b^c/1.3\approx 10^{0.778}$).

suprathreshold $X_b\leqslant X_b^c$. We suggest that both subthreshold and suprathreshold optimal systems are the appealing nonlinear signal processors in practical applications in the context of information transmission. According to the role of the noise, we can refer the subthreshold and some slightly suprathreshold optimal systems to the conventional aperiodic stochastic resonators, and regard other suprathreshold systems as the results of the nonlinear system optimization. It is shown that the conventional ASR effect requires a threshold to be set thus not making the system very robust for engineering applications [10]. The suprathreshold systems with $X_b<X_b^d-\epsilon$ are more applicable in the view of information transmission, but at the risk of not employing the positive role of noise.

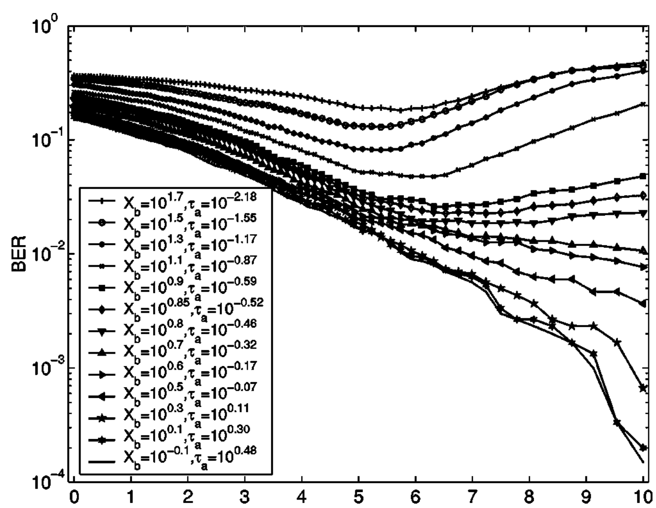


FIG. 4. Numerical results of the BER as a function of $D$ for the different systems shown in Fig. 3. Here, $A=3$, $T_b=4$, and $\Delta t=0.01\tau_a$.

### IV. CONCLUSION

In conclusion, two methods of realizing ASR phenomenon, i.e., the conventional way of adding noise and the method of tuning system parameters, have been compared detailedly. After a scale transformation, the conventional way of adding noise can be viewed as a specific case of the method of tuning system parameters, and the two methods can be compared in a parameter space geometrically. It has been demonstrated theoretically and numerically that when the systems are adjustable, especially as an algorithm in signal processing, tuning system parameters is more practical than adding noise to the nonlinear systems. Furthermore, the optimal systems searched by the tuning system parameter method were discussed and classified as subthreshold and suprathreshold. According to the role of the noise in the optimal system, we can refer the resonance phenomena realized by the subthreshold and slightly suprathreshold optimal systems to the concept of the conventional ASR. But, it is suitable to consider other suprathreshold optimal systems as the outcomes of the nonlinear system optimization, without utilizing the constructive role of the noise.

The conventional ASR has been gaining increasing interest as a potential signal-processing tool [5–11,13,15,20–25,32–42]. Similarly, it is meaningful to research the resonance phenomenon with tuning system parameters in the signal processing field. Some subjects are very promising and currently under study, for example, studying nonlinear system with input multiplicative noise and extending the nonlinear system to Hamiltonian or quasi-Hamiltonian systems.

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### APPENDIX A: SYSTEM RESPONSE SPEED

Rescale the variants of Eq. (2) as

$$t = \tau\tau_a, \quad x = y\sqrt{D/\tau_a}, \quad X_b = \overline{X}_b\sqrt{D/\tau_a}, \quad A = \overline{A}\sqrt{D/\tau_a}. \tag{A1}$$

Equation (2) becomes

$$\frac{\partial \rho(y,\tau)}{\partial \tau} = \frac{\partial}{\partial y}[V'(y)\rho(y,\tau)] + \frac{\partial^2\rho(y,\tau)}{\partial y^2}, \tag{A2}$$

where $V'(y) = -y + y^3/\overline{X}_b^2 \pm \overline{A}$. Assume

$$\rho(y,\tau) = p(y)\exp[-V(y)]\exp[-\lambda\tau], \tag{A3}$$

where $\lambda > 0$ and $p(y) \neq 0$. Substituting Eq. (A3) into Eq. (A2), we have

$$\lambda p(y)\exp[-V(y)] = -p''(y)\exp[-V(y)] + p'(y)V'(y)\exp[-V(y)]. \tag{A4}$$

Multiplying both side of Eq. (A4) by $p(y)$ and integrating it, Xu *et al.* [41] deduced its variational form as [47]

$$\lambda = \mathrm{st.}_{p(y)\neq 0} \times \frac{\int_{-\infty}^{+\infty}[-p''(y)p(y) + p'(y)p(y)V'(y)]\exp[-V(y)]dy}{\int_{-\infty}^{+\infty}p^2(y)\exp[-V(y)]dy}, \tag{A5}$$

where st. means the stationary value in the variational problem [47]. Since

$$\begin{aligned}\int_{-\infty}^{+\infty} -p''(y)p(y)\exp[-V(y)]dy &= -p(y)p'(y)\exp[-V(y)]\Big|_{-\infty}^{+\infty} + \int_{-\infty}^{+\infty}p'^2(y)\exp[-V(y)]dy - \int_{-\infty}^{+\infty}p'(y)p(y)V'(y)\exp[-V(y)]dy \\ &= \int_{-\infty}^{+\infty}p'^2(y)\exp[-V(y)]dy - \int_{-\infty}^{+\infty}p'(y)p(y)V'(y)\exp[-V(y)]dy,\end{aligned}$$

Eq. (A5) can be rewritten as

$$\lambda = \mathrm{st.}_{p(y)\neq 0}\frac{\int_{-\infty}^{+\infty}p'^2(y)\exp[-V(y)]dy}{\int_{-\infty}^{+\infty}p^2(y)\exp[-V(y)]dy}. \tag{A6}$$

Assume $p(y) = d_0 + d_1 y + \cdots + d_n y^n$, we have

$$([K] - \lambda[M])\{d\} = 0, \tag{A7}$$

where eigenvectors $\{d^i\} = [d_0^i, d_1^i, \ldots, d_n^i]$ are corresponding to eigenvalues $\{\lambda\} = [\lambda_0, \lambda_1, \ldots, \lambda_n]$, where $\lambda_i \leqslant \lambda_{i+1}$, $i = 0, 1, \ldots, n-1$. The elements of matrices $[M]$ and $[K]$ are

$$m_{ij} = \int_{-\infty}^{+\infty} y^{i+j}\exp[-V(y)]dy,$$

$$k_{ij} = \int_{-\infty}^{+\infty} ijy^{i+j-2}\exp[-V(y)]dy, \tag{A8}$$

where $i, j = 0, 1, 2, \ldots, n$. It has been demonstrated that the matrix $[M]$ is positive definite and the matrix $[K]$ is semi-positive definite [41]. Then, the minimum eigenvalue $\lambda_0$ is zero corresponding to the stationary solution of Eq. (A2). The minimum positive eigenvalue $\lambda_1$ is called the system response speed, which dominates the speed tending to the stationary state. Note the time scale transformation in Eq.

(A1); the real system response speed is $\overline{\lambda_1}=\lambda_1/\tau_a$.

## APPENDIX B. THE PROOF THAT THE BER INCREASES AS THE INPUT $\overline{A}$ DECREASES

After the transformation of Eq. (7), the quasi-stationary probability density of Eq. (5) can be rewritten as

$$\rho(y,\tau|s_i(\tau)) = C\exp[-\overline{\tau_a}G(y)], \tag{B1}$$

where $C$ is the normalization constant and $G(y)=-y^2/2+y^4/(4\overline{X_2^b})\pm\alpha\overline{A}y$. Since the decision time is the end time of $T_b$, the term $\tau$ of $\rho(y,\tau|s_i(\tau))$ is unity in terms of the transformation of Eq. (7). Thus, we have probabilities of error

$$P(1|0)=\int_0^{+\infty}\rho(y,1|s_0(\tau))dy,\quad P(0|1)=\int_{-\infty}^{0}\rho(y,1|s_1(\tau))dy. \tag{B2}$$

The first partial derivative of $P_e$ with respect to $\overline{A}$ is

$$\frac{\partial P_e}{\partial\overline{A}}=\frac{1}{2}\left[\frac{\partial P(0|1)}{\partial\overline{A}}+\frac{\partial P(1|0)}{\partial\overline{A}}\right] =\frac{1}{2}\left[\int_{-\infty}^{0}\alpha y\rho(y,1|s_1(\tau))dy+\int_0^{+\infty}-\alpha y\rho(y,1|s_0(\tau))dy\right] \leqslant 0. \tag{B3}$$

This indicates that the BER increases as the input $\overline{A}$ decreases. Moreover, it has been demonstrated in [41] that the system response speed becomes slower and slower as the input $\overline{A}$ reduces, and the coefficient $\alpha$ then decreases. Therefore, for given system parameters, we can deduce that the BER is a monotonic decreasing function of the input $\overline{A}$. Thus, the point ***d*** in Fig. 1 will provide a lower BER than the conventional resonance point ***b*** with the same system parameters $\overline{\tau_a}$ and $X_b$.


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